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OSCILLATIONS OF A ROTARY FLUID FLOWING OUT OF A CLOSED VESSEL

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Consideration is given to the problem of the proper motion of a rotary fluid that fills the entire cylindrical tank in the presence of outflow through a rigid bottom. The problem is solved in a quasistationary formulation within the framework of the ideal-fluid model with allowance for hydraulic losses in the flow of the fluid through the bottom of the vessel. The spectrum of eigenvalues is investigated and the characteristics of wave motions of the fluid are revealed; the results of calculating the wave numbers and the complex attenuation factor are given.

Formulation of the Problem. Let an ideal incompressible fluid closed with a cover fill a cylindrical vessel of radius R_0 to depth H and, in steady-state motion, rotate together with it about a fixed OX axis with a constant angular velocity ω_0 and flow out through drainage surface Σ with velocity V_Σ. We assume that when the fluid is flowing out the cover remains perpendicular to the axis of rotation at all times and any overflowing of the fluid through the cover is excluded. We denote the region occupied by the fluid by Q, a solid lateral wall by S, and a wetted surface of the cover by Γ. We introduce a moving cylindrical coordinate system $Oxr\eta$ tied to the cover, i.e., rotating with angular velocity ω_0 and traveling together with the cover with velocity V_0.

Let us consider the problem of small natural oscillations of the fluid in a quasistationary formulation, assuming that the region occupied by the fluid has no time to change substantially in the characteristic times of its investigated motions. Then, to determine the field $\vec{v} = \vec{v}(x, t)$ of the velocities of fluid particles relative to the steady-state motion, we have the following problem written in a moving reference system:

$$\frac{\partial \vec{v}}{\partial t} - 2\omega_0 (\vec{v} \times \vec{k}) + \nabla p = 0 , \tag{1}$$

$$\nabla \cdot \vec{v} = 0 \quad \text{in} \quad Q , \tag{2}$$

$$\vec{v} \cdot \vec{n} = 0 \quad \text{on} \quad S \cup \Gamma , \tag{3}$$

$$\vec{v} \cdot \vec{n}_\Sigma = \frac{p}{\gamma} \quad \text{on} \quad \Sigma , \tag{4}$$

$$\vec{v}(x, 0) = \vec{v}^{\,0} (x) . \tag{5}$$

Here $\vec{n}$ and $\vec{n}_\Sigma$ are the external normals to the surfaces $S \cup \Gamma$ and Σ, respectively; $\vec{k}$ is the unit vector directed along the Ox axis; p is the modified pressure $p = \frac{p'}{\rho}$; p' is the deviation of the pressure from the equilibrium value; $\gamma = \psi(V_\Sigma - V_0)$, where ψ is the coefficient of resistance of the drainage surface. The condition on the

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drainage surface is obtained based on linearization of the equation for a pressure difference on the drainage surface and is employed in calculating the dynamic characteristics of liquid-propellant rockets [1].

For the coefficient of resistance $\gamma = \infty$ (the absence of drainage), problem (1)-(5) is a problem of the motions of a fluid that fills the entire circular cylindrical vessel. Its mathematical aspects are studied in [2-5]. A spectral problem with numerical calculations of eigenfrequencies that corresponds to it is contained in [6-8].

Eliminating the variable $\vec{v}$, we will seek a solution in the form of the traveling waves $p(x, r, \eta, t) = P(x, r) \exp^{(im\eta - \Omega t)}$, $m = 0, \pm 1, \pm 2, \ldots$. Then, in the moving coordinate system, the spectral problem for the circular cylindrical vessel can be written in the form

$$\Omega^2 \left(\frac{\partial^2 P}{\partial r^2} + \frac{1}{r} \frac{\partial P}{\partial r} - \frac{m^2}{r^2} P + \frac{\partial^2 P}{\partial x^2} \right) + 4\omega_0^2 \frac{\partial^2 P}{\partial x^2} = 0 \quad \text{in} \quad Q, \tag{6}$$

$$\frac{\partial P}{\partial x} = 0 \quad \text{on} \quad \Gamma, \tag{7}$$

$$\Omega^2 \frac{\partial P}{\partial r} - 2\omega_0 \Omega i \frac{m}{r} P = 0 \quad \text{on} \quad S, \tag{8}$$

$$\frac{\partial P}{\partial x} + \Omega \frac{1}{\gamma} P = 0 \quad \text{on} \quad \Sigma, \tag{9}$$

$$m = 0, \pm 1, \pm 2, \ldots .$$

Here Ω is the complex attenuation factor of the wave motions of the fluid. If we assume $\operatorname{Im} \Omega > 0$ the number $m < 0$ will correspond to the waves traveling in the direction of fluid rotation, i.e., to forward waves, while the number $m > 0$ will correspond to backward waves. When $m = 0$ we have the case of standing waves.

Model Problems. Study of model problems in the absence of rotation ($\omega_0 = 0$) and in the absence of drainage ($\gamma = \infty$) shows that in the system there can exist two forms of wave motions – the internal waves due to the presence of rotation and the waves on the drainage surface that will subsequently be referred to as drainage waves. Unlike waves on a free surface, in the absence of drainage, drainage waves are motions aperiodic in time with the attenuation factor

$$\Omega_{mn} = \xi_{mn} \frac{\gamma}{R_0} \tanh \xi_{mn} \frac{H}{R_0},$$

where ξ_{mn} are the zeros of the function $dJ_m(\xi)/d\xi$; $J_m(\xi)$ is the mth order Bessel function of the first kind. The set of the real numbers $\{\Omega_{mn}\}_{n=1}^{\infty}$ forms a discrete spectrum with the limiting point $\Omega_{mn} \to \infty$ when $n,m \to \infty$ [9].

The internal waves in the rotary cylindrical vessel filled entirely without drainage are oscillations with the frequency

$$\omega_{mnl} = \frac{2\omega_0}{\sqrt{\left(\left(\frac{\xi_{mn} H}{l\pi} \right)^2 + 1 \right)}},$$

$$m = 0, \pm 1, \pm 2, \ldots, \quad n = 1, 2, 3, \ldots, \quad l = 1, 2, 3, \ldots .$$

The set of eigenfrequencies $\{\omega_{mnl}\}_{n,l=1;m=0}^{\infty}$ forms a limiting spectrum on the interval $[0, i2\omega_0]$, while the numbers ξ_{mnl} are determined from the equation (see, for example, [6, 7])

$$\frac{J_{m-1}(\xi_{mn})}{J_m(\xi_{mn})}=\frac{m}{\xi_{mn}}\left(1+\sqrt{\left(\left(\frac{\xi_{mn}H}{l\pi}\right)^2+1\right)}\right),\quad l=1,2,3,\dots .$$

Derivation of the Characteristic Equations of the Problem. With the method of separation of variables $P(x, r) = X(x)R(r)$ we obtain the system of characteristic equations for determination of the dimensionless wave numbers ζ and ξ:

$$\bar{\gamma}\tanh\zeta\bar{H}=\frac{1}{\sqrt{\xi^2-\zeta^2}},\tag{10}$$

$$\frac{J_{m-1}(\xi)}{J_m(\xi)}=\frac{m}{\xi}\left(1+\frac{i\sqrt{\xi^2-\zeta^2}}{\xi}\right),\quad \bar{H}=\frac{H}{R_0},\tag{11}$$

where

$$\zeta=\mu R_0;\quad \xi=kR_0;\quad \lambda=\frac{\Omega}{2\omega_0};\quad \bar{\gamma}=\frac{\gamma}{2\omega_0R_0};$$

R_0 is the characteristic dimension of the vessel; $J_m(\xi)$ is the mth order Bessel function of the first kind, while the eigenvalue λ is related to the wave numbers ξ and ζ by the formula

$$\lambda=\frac{\zeta}{\sqrt{\xi^2-\zeta^2}}.\tag{12}$$

The system of transcendental equations (10)-(11) with prescribed $\bar{\gamma}$ and $\bar{H}$ will be solved for different numbers $m = 0, \pm1, \pm2, \dots$ that determine the oscillation modes of the considered hydromechanical system.

By the mode of the mth order oscillations we will mean the set of dimensionless wave numbers ξ and ζ and the eigenvalue λ together with the related eigen- and associated functions.

Investigation of Transcendental Equations. In the system of equations (10)-(11), the integral functions $\tanh\zeta\bar{H}$ and $f_1(\xi) = J_{m-1}(\xi)/J_m(\xi)$ are transcendental meromorphic functions of the complex variables ξ and ζ, while the function $f_2(\xi, \zeta) = \sqrt{\xi^2-\zeta^2}$ is an irrational two-valued function with possible branch points $\zeta = \pm\xi$. Consequently, we can assume that system (10)-(11) will have the complex solutions $\zeta = \zeta^{(r)} + i\zeta^{(i)}$ and $\xi = \xi^{(r)} + i\xi^{(i)}$.

First of all, it is easy to show that the system of equations (10)-(11) with no $m = 0, \pm1, \pm2, \dots$ has the following solutions:

$$\text{a) } \xi=i\xi_0,\quad \zeta=i\zeta_0,\quad \xi_0,\zeta_0\in\mathbb{R};$$

$$\text{b) } \xi=i\xi_0,\quad \zeta=\zeta_0,\quad \xi_0,\zeta_0\in\mathbb{R};$$

$$\text{c) } \xi=\xi_0,\quad \zeta=i\zeta_0,\quad \xi_0,\zeta_0\in\mathbb{R}.$$

For a more complete investigation of transcendental equations (10)-(11), in what follows we will consider the cases of standing waves ($m = 0$) and traveling waves ($m \neq 0$) individually.

In the case of standing waves, upon separation of the variables from Eq. (6) we obtain the classical Neumann problem for the Bessel function

$$-\left(R'' + \frac{1}{\bar{r}} R'\right) = \xi_0^2 R\,, \quad R'(\xi_0 \bar{r}) = 0\,, \quad \bar{r} = 1\,, \tag{13}$$

where $R(\xi_0 \bar{r}) = J_0(\xi_0 \bar{r})$. Nontrivial solutions of problem (13) correspond to the discrete positive set of the numbers $\{\xi_{0n}\}_{n=1}^{\infty}$ with a unique limiting point at infinity and the asymptotic behavior of the numbers ξ_{0n}[10]

$$\xi_{0n} \sim \frac{\pi}{4} + \pi n + O\left(\frac{1}{n}\right), \quad n \to \infty\,.$$

Thus, to study the properties of the roots of characteristic equations (10)-(11), in the case of standing waves it will suffice to consider the first transcendental equation

$$\tanh \zeta \bar{H} = \frac{\sigma}{\sqrt{\xi_{0n}^2 - \zeta^2}}\,, \quad \xi_{0n} \in R\,, \quad \sigma = \frac{1}{\gamma}\,. \tag{14}$$

First, let us consider ζ to be a real number. Accordingly, we denote $f_1(\zeta) = \tanh \zeta\bar{H}$ and $f_2(\zeta) = \sigma/\sqrt{\xi_{0n}^2 - \zeta^2}$, $0 \le \zeta < \xi_{0n}$. By direct computation of the derivatives we establish that the curve $f_1(\zeta)$ is concave ($f_1''(\zeta) > 0$) and the curve $f_2(\zeta)$ is convex ($f_2''(\zeta) < 0$) $\forall\ \zeta$, $0 \le \zeta < \xi_{0n}$. Consequently, the curves $f_1(\zeta)$ and $f_2(\zeta)$ can intersect at no more than two points. At the point of tangency of the curves, we have one double real root. If the curves $f_1(\zeta)$ and $f_2(\zeta)$ do not intersect and do not touch each other, Eq. (14) has two complex conjugate roots. The proof of the latter fact is similar to the proof given in [11, 12].

It turns out that Eq. (14) for the same $\sigma, \bar{H}$, and ξ_{0n}, for which it has two real or two complex conjugate roots, also has an infinite set of simple complex roots $\{\zeta_{0nl}\}_{l=1}^{\infty}$ at any fixed number n. To prove this fact, we resort to the resolution of the meromorphic function $f_1(\zeta)$ into partial fractions

$$\tanh \zeta\bar{H} = \sum_{k=1}^{\infty} \frac{2\zeta\bar{H}}{\frac{(2k-1)^2}{4}\pi^2 + (\zeta\bar{H})^2}$$

and set up the approximating equation

$$F^L(\zeta) = f_1^L(\zeta) - \frac{\sigma}{\sqrt{\xi_{0n}^2 - \zeta^2}} = 0\,,$$

where

$$f_1^L(\zeta) = \sum_{k=1}^{L} \frac{2\zeta\bar{H}}{\frac{(2k-1)^2}{4}\pi^2 + (\zeta\bar{H})^2}\,.$$

Let $\zeta = \zeta^{(r)} + i\zeta^{(i)}$; having represented the approximating function $F^L(\zeta)$ in the form

$$F^L(\zeta) = u^L(\zeta^{(r)}, \zeta^{(i)}) + iv^L(\zeta^{(r)}, \zeta^{(i)})\,,$$

we obtain the approximating system of equations

$$u^L(\zeta^{(r)}, \zeta^{(i)}) = 0\,, \quad v^L(\zeta^{(r)}, \zeta^{(i)}) = 0\,. \tag{15}$$

Let us resort to a graphical solution of the system.

In Fig. 1, in the plane of the variables $\zeta^{(r)}$ and $\zeta^{(i)}$ for the case where there are two real roots in the solution of Eq. (14), geometric images of the equations $u^L(\zeta^{(r)}, \zeta^{(i)}) = 0$ (the solid line) and $v^L(\zeta^{(r)}, \zeta^{(i)}) = 0$ (the

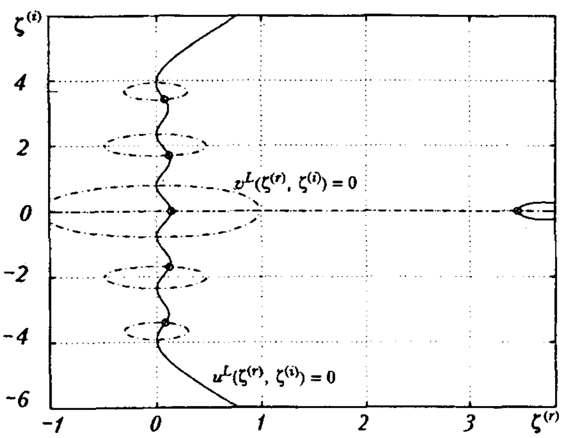

Fig. 1. Graphical solution of system (15).

dash–dot line) are constructed at $L = 3$. Their intersection points yield the sought roots ζ_{0nl} of system (15) for each fixed number n. As is seen from the graphical solution the complex roots ζ_{0ln} are grouped along the imaginary axis; $\zeta^{(r)} > 0$. Because of the convergence of the series $\sum_{k=1}^{\infty} \frac{2\zeta\overline{H}}{\frac{(2k-1)^2}{4}\pi^2 + (\zeta\overline{H})^2}$ the functions $f_1^L(\zeta)$ will converge to the function $f_1(\zeta)$ when $L \to \infty$, while the functions $F^L(\zeta)$ will converge to the function $F(\zeta) = \tanh \zeta\overline{H} - \frac{\sigma}{\sqrt{\zeta_{0n}^2 - \zeta^2}}$. According to the Hurwitz theorem [13], the roots of Eq. (14) will be a limit of the roots of the equation $F^L(\zeta) = 0$ when $L \to \infty$.

In the case of $m = 1$, we will write system (10)-(11) as

$$\overline{\gamma} \tanh \zeta\overline{H} = \frac{1}{\sqrt{\xi^2 - \zeta^2}}, \tag{16}$$

$$\frac{J_0(\xi)}{J_1(\xi)} = \frac{1}{\xi}\left(1 + \frac{i\sqrt{\xi^2 - \zeta^2}}{\zeta}\right). \tag{17}$$

Employing the relations for the Bessel functions, we rewrite (17) in the form

$$\frac{J_2(\xi)}{J_1(\xi)} = \frac{1}{\xi}\left(\frac{i\sqrt{\xi^2 - \zeta^2}}{\zeta} - 1\right)$$

and as in the case $m = 0$ we reduce the system to one equation

$$F(\zeta) = 0, \tag{18}$$

where

$$F(\zeta) = \frac{J_2\left(\sqrt{\left(\frac{\sigma^2}{\tanh^2 \zeta\overline{H}} + \zeta^2\right)}\right)}{J_1\left(\sqrt{\left(\frac{\sigma^2}{\tanh^2 \zeta\overline{H}} + \zeta^2\right)}\right)} - \frac{1}{\sqrt{\left(\frac{\sigma^2}{\tanh^2 \zeta\overline{H}} + \zeta^2\right)}}\left(\frac{i\sigma}{\zeta \tanh \zeta\overline{H}} - 1\right). \tag{19}$$

Unlike the case $m = 0$, where n was fixed and the complex roots ζ_{0nl} determined from formula (14) formed in the vicinity of the imaginary axis, for $m = 1$ the parameters ξ are computed upon finding the numbers ζ from (19). In this case, there is an infinite set of complex roots $\{\zeta_{1nl}\}_{l=1,\,n=1}^{\infty}$ that form both along the imaginary and real axes of the plane of the complex variable ζ.

To prove this fact, we resort to the resolution of the Bessel function into partial fractions [10, p. 549]

$$\frac{J_2(\xi)}{J_1(\xi)} = \xi \sum_{k=1}^{N} \frac{2}{\xi_{mk}^2 - \xi^2}$$

and set up the approximating system of equations

$$u^N(\zeta^{(r)}, \zeta^{(i)}) = 0\,, \quad v^N(\zeta^{(r)}, \zeta^{(i)}) = 0\,. \tag{20}$$

Using the same line of reasoning as in the case $m = 0$, we see that Eq. (18) has the infinite set of complex roots $\{\zeta_{1nl}\}_{l=1,\,n=1}^{\infty}$.

Eigenvalues and Eigenfunctions. Upon determination of the wave numbers $\zeta = \zeta^{(r)} + i\zeta^{(i)}$, using formula (12), we can find the eigenvalues λ_{mnl} and eigenfunctions that correspond to them and determine the oscillation modes. To identify the oscillation modes, we resort to the asymptotic form of large indexes in the case of both standing and traveling waves.

Asymptotic form of large indexes at $m = 0$. We consider the asymptotic behavior of the roots of Eq. (14). Let n be fixed (ξ_{0n} = const). With large l the solution of Eq. (14) will be sought in the form

$$\zeta_{0nl} = i\frac{l\pi}{\bar{H}} + i\varepsilon_l\,, \quad i = \sqrt{-1}\,, \quad \varepsilon_l \to 0\,, \quad l \to \infty\,. \tag{21}$$

Having substituted (21) into (14), upon simple manipulations we obtain the asymptotic expression for the roots of (14)

$$\zeta_{0nl} = i\frac{\pi l}{\bar{H}}\left(1 - \frac{\bar{H}}{\gamma\pi^2 l^2}\right) + \frac{1}{\gamma l\pi} + o(l^{-3})\,, \quad l \to \infty\,. \tag{22}$$

Let l be fixed now ($l = -1$, $l = -2$). When $n \to \infty$ ($\xi_{0n} \to \infty$) the root $\zeta_{0,n,-1} \to \xi_{0n}$ ($n \to \infty$) and $\zeta_{0,n,-2}$ tends to 0. Therefore we will seek $\zeta_{0,n,-1}$ and $\zeta_{0,n,-2}$ when $n \to \infty$ in the form

$$\zeta_{0,n,-1} = \xi_{0n} - \varepsilon_{n,-1}\,, \quad \zeta_{0,n,-2} = 0 + \varepsilon_{n,-2}\,.$$

Having substituted these values into (14) and having determined $\varepsilon_{n,-1}$ and $\varepsilon_{n,-2}$, we obtain the asymptotic expression of the roots of Eq. (14) when $n \to \infty$:

$$\zeta_{0,n,-1} = \xi_{0n} - \frac{\sigma^2}{2\xi_n} + o(\xi_{0n}^{-2})\,, \quad n \to \infty\,, \tag{23}$$

$$\zeta_{0,n,-2} = \frac{\sigma}{\bar{H}\xi_{0n}} + o(\xi_{0n}^{-3})\,, \quad n \to \infty\,. \tag{24}$$

When ζ_{0nl} and ξ_{0n} are known the eigenvalues λ_{0nl} and hence the complex attenuation factor $\Omega_{0nl} = 2\omega_0\lambda_{0nl}$ of initial spectral problem (6)-(9) can be determined by the formula

$$\lambda_{0nl} = \left(\frac{\zeta_{0nl}^2}{\zeta_{0nl}^2 - \xi_{0n}^2} \right), \quad l = -1, -2, \quad n = 1, 2, 3, \dots . \tag{25}$$

The eigenvalues λ_{0nl} correspond to the eigenfunctions of problem (6)-(9)

$$P_{0nl}(x, r) = J_0 \left(\xi_{0n} \frac{r}{R_0} \right) \cosh \zeta_{0nl} \frac{x}{R_0} . \tag{26}$$

Let us show that the eigenvalues λ_{0nl} (l = 1, 2, 3, ..., n = 1, 2, 3, ...) correspond to wave motions in the rotary fluid with the maximum intensity deep within the fluid, i.e., the internal waves. For this purpose, we resort to asymptotic expression (22) for the wave numbers ζ_{0nl} ($l \to \infty$, n is fixed). Having substituted ζ_{0nl} into the expression for the eigenvalue λ, we obtain

$$\lambda_{0nl} = \sqrt{\left(\frac{\left[i \frac{\pi l}{\overline{H}} \left(1 + \frac{\overline{H}^2}{\overline{\gamma}^2 \pi^4 l^4} \right) + \frac{1}{\overline{\gamma} l \pi} \right]^2}{\xi_{0n}^2 - \left[i \frac{\pi l}{\overline{H}} \left(1 + \frac{\overline{H}^2}{\overline{\gamma}^2 \pi^4 l^4} \right) + \frac{1}{\overline{\gamma} l \pi} \right]^2} \right)} =$$

$$= \sqrt{\left(\frac{-\frac{\pi^2 l^2}{\overline{H}^2} \left(1 - 2 \frac{\overline{H}}{\overline{\gamma} \pi^2 l^2} (i + 1) + o(l^{-4}) \right)}{\xi_{0n}^2 + \frac{\pi^2 l^2}{\overline{H}^2} \left(1 - 2 \frac{\overline{H}}{\overline{\gamma} \pi^2 l^2} (i + 1) + o(l^{-4}) \right)} \right)} =$$

$$= i \overset{*}{\omega}_{0nl} (1 + o(l^{-2})) , \quad l \to \infty , \quad n = 1, 2, 3, \dots ,$$

where $\omega^*_{onl} = \left(\frac{\pi^2 l^2}{\overline{H}^2} / \xi_{0n}^2 + \frac{\pi^2 l^2}{\overline{H}^2} \right)^{1/2}$ is the dimensionless frequency of natural oscillations of the rotary fluid in a cylindrical vessel with circular cross section without outflowing and with complete filling. As the asymptotic representation for the wave numbers ζ_{0nl} and expression (26) yield, these wave motions are described by the eigenfunctions

$$P_{0nl}(x, r) = J_0 \left(\xi_{0n} \frac{r}{R_0} \right) \cos \frac{\pi l}{\overline{H}} x .$$

We now direct our attention to asymptotic expressions for the wave numbers ζ_{0nl} ($l = -1, -2$; $n \to \infty$). We will show that the eigenvalues λ_{0nl}, in this case, will correspond to the wave motions of the fluid with the maximum intensity on the drainage surface. We consider first the situation where $l = -1$. Having substituted ζ_{0nl} from (23) into (25), we obtain

$$\lambda_{0,n,-1}=\sqrt{\left(\frac{\left(\xi_{0n}-\dfrac{\sigma^2}{2\xi_{0n}}+o\,(\xi_{0n}^{-2})\right)^2}{\xi_{0n}^2-\left(\xi_{0n}-\dfrac{\sigma^2}{2\xi_{0n}}+o\,(\xi_{0n}^{-2})\right)^2}\right)}=$$

$$=\sqrt{\left(\xi_{0n}^2\left(\frac{1}{\sigma^2}-o\,(\xi_{0n}^{-2})\right)\right)}=\frac{\xi_{0n}}{\sigma}+o\,(\xi_{0n}^{-1}),\quad n\to\infty, \tag{27}$$

i.e., the dominant term of asymptotic form (27) coincides with the dominant term of the asymptotic form for drainage waves [13]. The solution $P_{0,n,-1}(x, r)$, in this case, permits the conclusion of the scale of fluid motion decreasing with distance upward from the drainage surface. Thus, the eigenvalue obtained by formula (27) at $l = -1$ and the eigenfunction at $l = -1$ determine the drainage waves. We now consider ζ_{0nl} at $l = -2$. We will show that in this case, too, the eigenvalue $\lambda_{0,n,-2}$ and the eigenfunctions $P_{0,n,-2}$ will also determine the drainage waves but with another waves number. Having substituted (24) into formulas (25) and (26), we have

$$\lambda_{0,n,-2}=\sqrt{\left(\frac{\left[\left(\dfrac{\sigma}{\overline{H}\xi_{0n}}\right)+o\,(\xi_{0n}^{-3})\right]^2}{\xi_{0n}^2-\left[\left(\dfrac{\sigma}{\overline{H}\xi_{0n}}\right)+o\,(\xi_{0n}^{-3})\right]^2}\right)}=\left(\frac{\sigma}{\overline{H}\xi_{0n}^2}\right)+o\,(\xi_{0n}^{-4}),\quad n\to\infty,$$

$$P_{0,n,-2}(x,r)=J_0\left(\xi_{0n}\frac{r}{R_0}\right)\cosh\left[\left(\frac{\sigma}{\overline{H}\xi_{0n}}+o\,(\xi_{0n}^{-3})\right)\right]\frac{x}{R_0},\quad n\to\infty. \tag{28}$$

By employing eigenfunction (28) it is easy to show that the scale of fluid motions decreases with distance upward from the drainage surface and hence solution (28) also describes the drainage waves.

Asymptotic form of large indexes at $m = 1$. Using the asymptotic form of large indexes $n\to\infty$, we can also obtain the asymptotic expressions for the roots ξ_{1nl} and ζ_{1nl}, which turn out to be equal to

$$\xi_{1nl}=\pi\left(n-\frac{1}{4}\right)-\frac{2}{\overline{\gamma}^2n^2\pi^2}-i\left(\frac{1}{\overline{\gamma}n^2\pi^2}-\frac{1}{\overline{\gamma}^2n^5\pi^5}\right)=\xi_{1nl}^{(r)}+i\xi_{1nl}^{(i)}, \tag{29}$$

$$\zeta_{1nl}=\pi\left(n-\frac{1}{4}\right)-i\frac{1}{\overline{\gamma}n^2\pi^2}=\zeta_{1nl}^{(r)}+i\zeta_{1nl}^{(i)}. \tag{30}$$

The eigenvalues ξ_{1nl} and ζ_{1nl} correspond to the eigenfunctions of problem (6)-(9)

$$P_{1nl}(x,r)=J_1\left(\xi_{1nl}\frac{r}{R_0}\right)\cosh\zeta_{1nl}\frac{x}{R_0}=$$

$$=J_1\left(\xi_{1nl}\frac{r}{R_0}\right)\cosh\zeta_{1nl}^{(r)}\frac{x}{R_0}\left(\cos\zeta_{1nl}^{(i)}\frac{x}{R_0}+i\tanh\zeta_{1nl}^{(r)}\frac{x}{R_0}\sin\zeta_{1nl}^{(i)}\frac{x}{R_0}\right). \tag{31}$$

Expression (31) and asymptotic formulas (29) and (30) show that the scale of fluid motion decreases with distance upward from the drainage surface and hence the solution that describes the eigenvalues $\lambda_{1n,-1}$ and $\lambda_{1n,-2}$ describe the drainage waves.

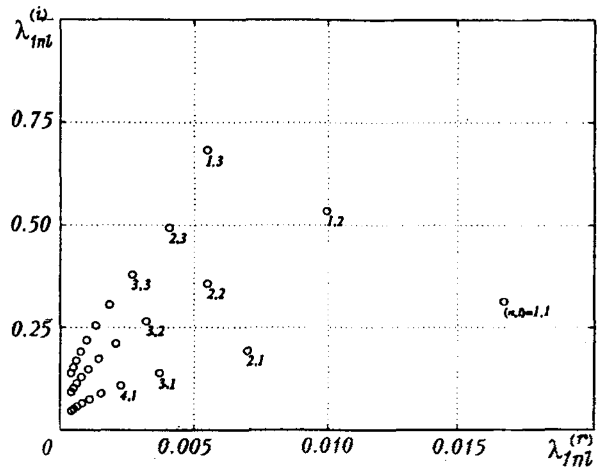

Fig. 2. Arrangement of the eigenvalues λ that correspond to internal waves on a complex plane. $m = 1$, $\overline{H} = 2$, and $\overline{\gamma} = 1$.

TABLE 1. Results of Computation of ζ_{1nl}, ξ_{1nl}, and λ_{1nl} at $m = 1$, $\overline{H} = 2$, and $\overline{\gamma} = 1$.

ξ_{1nl}	n	l	ζ_{1nl}	λ_{1nl}
2.005093–0.363113i	1	–1	1.750099–0.417056i	1.749649–0.420305i
3.839011–0.133749i		–2	0.133214+0.004876i	0.034634+0.002480i
5.331849–0.037131i	2	–1	5.237239–0.037802i	5.237239–0.037802i
7.016704–0.071854i		–2	0.071742+0.000744i	0.010222+0.000210i
8.536351–0.014011i	3	–1	8.477576–0.014108i	8.477576–0.014108i
10.173829–0.049342i		–2	0.049304+0.000240i	0.004846+0.000047i
4.752236–0.025349i	1	1	0.101254+1.571957i	0.016723+0.314282i
4.970592–0.006356i		2	0.085858+3.142355i	0.009942+0.534485i
5.046163–0.002229i		3	0.072924+4.712939i	0.005479+0.682624i
7.971489–0.016563i	2	1	0.061854+1.571013i	0.006941+0.193388i
8.209625–0.004750i		2	0.057129 +3.141755i	0.005488+0.357435i
8.297002–0.001929i		3	0.052593+4.712542i	0.004080+0.493894i
11.143644–0.012224i	3	1	0.044547+1.570869i	0.003731+0.139592i
11.389063–0.003671i		2	0.042422 +3.141646i	0.003257+0.265921i
11.481040–0.001583i		3	0.040375+4.712443i	0.002739+0.379718i

Computation of the Roots of the System of Transcendental Equations. The numerical solution of system (10)-(11) is obtained with the Newton method of successive iteration. Upon finding the roots of ζ and ξ the eigenvalues λ are computed by formula (12).

The diagram in Fig. 2 shows the distribution of the complex attenuation coefficient λ of the internal waves of nonaxisymmetric oscillations ($m = 1$) on a complex surface. It is seen that the spectrum is a discrete set of roots with the accumulation points on the interval [0i; 1i].

Table 1 shows values of the wave numbers ξ and ζ and the eigenvalue λ at $n = 1$, 2, and 3 and $l = -2, -1$, 1, 2, and 3.

Conclusions. The investigation performed permits the following conclusions.

1. The set of normal modes consists of internal waves and drainage waves. The internal waves are characteristic of a rotary fluid and occur throughout its entire volume. The drainage waves are generated on the drainage surface; the scale of these motions decreases with distance deep into the fluid.

2. Both the internal and drainage waves are oscillatory motions attenuating in time. The presence of the drainage surface transforms the internal waves from oscillatory motions with a constant amplitude and a limiting spectrum to decaying oscillations with a discrete spectrum, while rotation transforms aperiodic motions on the drainage surface to decaying oscillations and generates an additional set of drainage waves with the limiting point of the attenuation factor at 0.

3. For standing waves ($m = 0$), there can exist such a relation of the rotational velocity ω_0 to the resistance to drainage γ, for which the drainage waves are purely aperiodic motions.

4. In the case of traveling waves, the presence of both forward waves propagating in the direction of rotation and backward waves is possible. The frequency of the backward waves is higher than the frequency of oscillations of the forward waves of the corresponding tone.

NOTATION

Q, region occupied by the fluid; Γ, wetted surface of the cover; S, solid lateral wall; Σ, drainage surface; $\vec{v}$, velocity field of fluid particles; p, modified pressure; R_0 and H, radius and depth of the vessel; γ, coefficient of resistance to drainage; ρ, density of the fluid; λ, eigenvalue of the problem – the complex attenuation factor; ζ and ξ, dimensionless wave numbers; V_0, descent velocity of the cover; V_Σ, rate of discharge of the fluid through the drainage surface Σ; Im Ω, imaginary part of the number Ω; $\mathbb{R}$, set of the real numbers. Subscripts: L and N, numbers of the terms of resolution in the corresponding finite sums of approximating expressions; (r) and (i), real and imaginary parts of the number.

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